Holomorphic functions

Def: $f: \underset{\neq \emptyset}{\Omega}^{\subset \mathbb{C}} \longrightarrow \mathbb{C}$ is holomorphic if it is complex-differentiable at any point $z \in \Omega$: $\exists \; f'(z) := \lim\limits_{\substack{h \to 0 \\ \in \mathbb{C}}} \dfrac{f(z+h) - f(z)}{h} \in \mathbb{C}$

Equivalent concepts:

- $f: \Omega \to \mathbb{C}$ is analytic: $\forall z_0 \in \Omega \; \exists U = D(z_0, \varepsilon) \subset \Omega$ s.t. $\forall z \in U$,
$f(z) = \sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$ for some $a_n \in \mathbb{C}$ $\left(a_n = \dfrac{f^{(n)}(z_0)}{n!} \right)$
Taylor series

- Cauchy-Riemann equations: $f: \Omega \to \mathbb{C}$ differentiable (over $\mathbb{R}$).
$f = u + iv \quad z = x + iy$. $\Leftrightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$

- Conformal maps: $f: \Omega \to \mathbb{C}$ is conformal if it preserves (oriented) angles (locally, infinitesimally). f Conformal $\Leftrightarrow$ f holomorphic and $f'(z) \neq 0 \; \forall z \in \Omega$.

- Integrals: $\gamma: \underset{[0,1]}{I} \to \Omega$ a piecewise C^1 path, $f: \Omega \to \mathbb{C}$ continuous
$\int_\gamma f(z)\,dz := \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt$.
f is holomorphic $\Leftrightarrow$ $\forall$ triangle $\underset{T}{\overset{\triangle}{ABC}} \subseteq \Omega$, $\int_{\gamma = \partial T} f(z)\,dz = 0$.

Properties of holomorphic functions

~~Analytic continuation:~~

Isolated zeros: $\underset{g}{f}: \Omega \to \mathbb{C}$, Ω connected, open.

If $\{z \in \Omega \mid \underset{g(z)}{f(z)} = 0\}$ has an accumulation point in Ω, then $f \equiv 0$. $(f \equiv g)$

Cauchy integral formula: $f : \Omega \to \mathbb{C}$ holomorphic, $p \in \Omega$

$$f^{(n)}(p) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-p)^{n+1}}\, dz,$$

for any loop γ ~~...~~ boundary of a small disc containing p and positively oriented.

Consequence:

$$f(p) = \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{it})\, dt \quad \forall r \text{ so that } \overline{D(p,r)} \subseteq \Omega \text{ (Mean value Thm)}$$

Maximum modulus principle.

$f : \Omega \to \mathbb{C}$ holomorphic on $\Omega \subseteq \mathbb{C}$ connected $(\neq \emptyset)$.

If $|f|$ admits a local maximum at $p \in \Omega$, then f is constant.

Open mapping theorem. $f : \Omega \to \mathbb{C}$ holomorphic,

If f is not locally constant (not constant on any open non-empty, or equivalently on any connected component), then f is an open map ($\forall U \subseteq \Omega$ open, $f(U)$ is open).

Def: An entire function is a holomorphic function $f : \mathbb{C} \to \mathbb{C}$.

Thm (Liouville). Any entire bounded function is constant.

* Rouché's theorem.

$f, g : U \to \mathbb{C}$, $U \subset \mathbb{C}$ simply connected. $K \subset U$ compact bounded by a closed path $\gamma = \partial K$. If $|f(z) - g(z)| < |g(z)|$ $\forall z \in \partial K$, then

$\# \text{Zeros}(f|_K) = \# \text{Zeros}(g|_K)$.

counted with multiplicity.

Singularities of holomorphic functions

$f : D^* \to \mathbb{C}$ holomorphic, $D = D(p, \varepsilon_0)$, $D^* = D \setminus \{p\}$.
$\hat{\mathbb{C}}$ $\vee$ 0

(p is an isolated singularity) p is a:

- removable singularity: if there exists $F : D \to \mathbb{C}$ holomorphic, s.t. $F|_{D^*} \equiv f$. $\Leftrightarrow$ $\exists \lim_{z \to p} f(z) \in \mathbb{C}$.   Ex: $f(z) = \frac{\sin z}{z}$

- pole singularity: if it is not removable, and $\exists m \in \mathbb{N}^*$ s.t. $(z-p)^m f(z)$ has a removable singularity at p. $\Leftrightarrow$ $\lim_{z \to p} f(z) = +\infty$ Ex: $f(z) = \frac{1}{z(e^z - 1)}$

$\Leftrightarrow \exists c_1 \ldots c_m \in \mathbb{C}$, $f(z) = \sum_{j=1}^{m} \frac{c_j}{(z-p)^j} + g(z)$ $\mathrm{res}(f, 0) = -\frac{1}{2}$ $m = 2$.

holomorphic, $g : D \to \mathbb{C}$.

Def: The minimal such m is the order of pole of f at p

c_1 is called the residue of f at p, denoted $\mathrm{res}(f, p)$

- essential singularity: if it is not removable, nor a pole.

$\Leftrightarrow$ $\nexists \lim_{z \to p} f(z)$ in $\hat{\mathbb{C}}$ (nor of $|f(z)|$ in $\mathbb{R}_+ \cup \{\infty\} = [0, +\infty]$)

Casorati-Weierstrass theorem
$\Leftrightarrow$ $\forall \varepsilon$, $0 < \varepsilon < \varepsilon_0$, $f(D(p, \varepsilon))$ is dense in $\mathbb{C}$

$\Leftrightarrow$ "   avoids at most one value in $\mathbb{C}$.
Great Picard theorem   Ex: $f(z) = e^{\frac{1}{z}}$, avoids the value 0.   (always the same of course)

Def: $\Omega \subseteq \mathbb{C}$ open non-empty set. A meromorphic map is a holomorphic map $f : \Omega \setminus S \to \mathbb{C}$, where S is a discrete set (all points in S are isolated), and $\forall p \in S$, f has not an essential singularity at p. (hence, f has only removable or pole singularities)

Residue's theorem. $\Omega \subset \mathbb{C}$ simply connected open set.

$f: \Omega \to \mathbb{C}$ meromorphic map. $S = \mathrm{Sing}(f)$. Let $\gamma: I \to \Omega \setminus S$ be a loop in Ω not passing through the singularities of f. Then,

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{p \in S} \mathrm{res}(f,p) \cdot \mathrm{ind}_\gamma(p),$$

Where $\mathrm{Ind}_\gamma(p) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-p} \in \mathbb{Z}$.

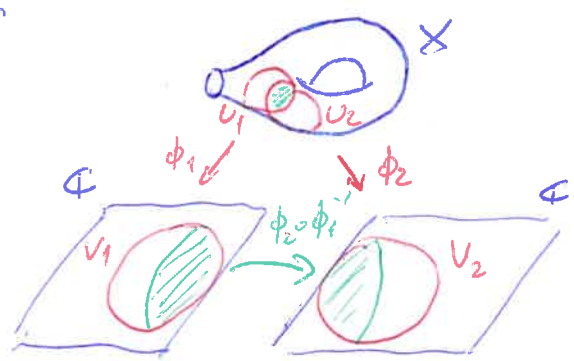
(✗)

Riemann surfaces.

Def: A Riemann surface X is a connected complex analytic manifold of complex dimension 1:

- X is a connected Hausdorff space.
- $\forall p \in X$, $\exists U \subset X$ open neighborhood of p, and $\phi: U \to V \subseteq \mathbb{C}$ homeomorphism satisfying the following property:

For any such $U_j \xrightarrow{\phi_j} V_j$ $j=1,2$, so that $U_1 \cap U_2 \neq \emptyset$, the map $\phi_2 \circ \phi_1^{-1}: \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is a biholomorphism (holomorphic and bijective $\Leftrightarrow$ holomorphic with holomorphic inverse).



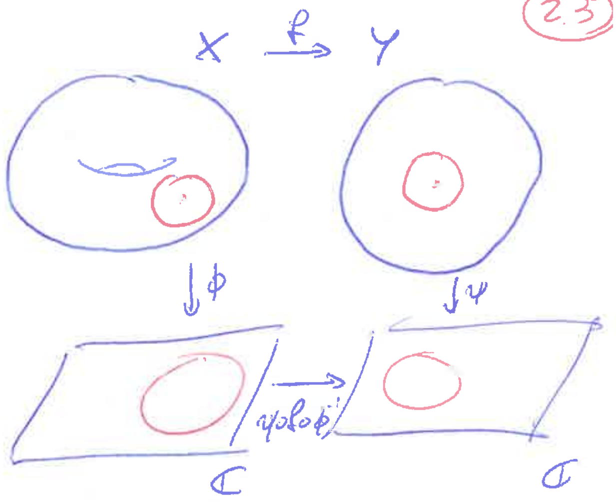
The maps ϕ are called "coordinate charts" (or local uniformising parameter), centered at p if $\phi(p) = 0$.

The maps $\phi_2 \circ \phi_1^{-1}$ are called "transition maps".

The family $\{(U_\alpha, \phi_\alpha)\}$ of charts satisfying the conditions above is called an "atlas".

Two atlases are compatible if their union is an atlas.
One can always take a maximal atlas.

A map $f: X \to Y$ between two Riemann surfaces X, Y is holomorphic if (it is continuous and) $\forall p \in X$, there are coordinate charts $\phi: U \to V$ of p and $\psi: U' \to V'$ of $f(p)$ such that $\psi \circ f \circ \phi^{-1}$ is holomorphic (in a neighborhood of $\phi(p)$).



f is a biholomorphism if it is holomorphic and bijective (by the open mapping theorem, f^{-1} is itself holomorphic).

Two Riemann surfaces X, Y are biholomorphic (or conformally isomorphic) if $\exists \phi: X \to Y$ biholomorphism. (We write $X \simeq Y$).

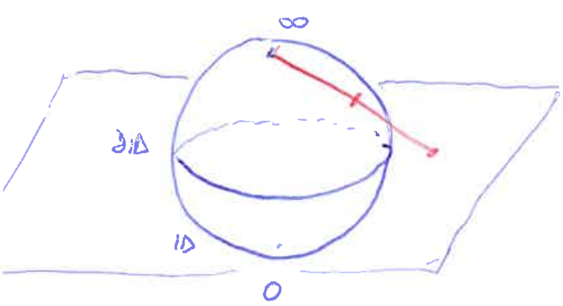
Examples:

$\mathbb{C}$, $\mathbb{D}$, any connected open subset U are Riemann surfaces (with atlas a unique coordinate chart, given by the natural inclusion $i: U \hookrightarrow \mathbb{C}$.

The Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, with two charts: $(\mathbb{C}, \phi_0)$, $(\hat{\mathbb{C}}\setminus\{0\}, \phi_\infty)$

where $\phi_0: \mathbb{C} \to \mathbb{C}$, $z \mapsto z$ and $\phi_\infty: \hat{\mathbb{C}}\setminus\{0\} \to \mathbb{C}$, $z \mapsto \frac{1}{z}$ $(\infty \mapsto 0)$. In this case the transition map is $z \mapsto \frac{1}{z}$, $\mathbb{C}^* \to \mathbb{C}^*$.

Remk: by stereographic projection, $\hat{\mathbb{C}}$ is homeomorphic to a sphere S^2.



The stereographic projection from the north pole N (identified with ∞) allow to identify $S^2 \setminus \{N\}$ with $\mathbb{C}$.

Remk: If Y is a Riemann surface and $f: X \to Y$ is a homeomorphism, then there exists a Riemann surface structure on X so that f is a biholomorphism. We say that such structure is induced by the one in Y. More generally, this applies to coverings.

Uniformisation theorem (difficult): Any simply connected Riemann surface is biholomorphic to either:

the Riemann sphere $\hat{\mathbb{C}}$, the plane $\mathbb{C}$, the disc $\mathbb{D}$.

Complex
Projective line

$$\mathbb{P}_1(\mathbb{C}) = \frac{\mathbb{C}^2 \setminus \{0\}}{\sim} \quad \text{where } (z_0, z_1) \sim (w_0, w_1) \text{ if } \exists \lambda \in \mathbb{C}^* \text{ s.t. } (w_0, w_1) = \lambda (z_0, z_1)$$

or $\mathbb{C}\mathbb{P}^1$

We denote by $[z_0 : z_1]$ the class represented by (z_0, z_1)

$\mathbb{P}_1(\mathbb{C})$ is a Riemann surface:

$$U_0 = \{z_0 \neq 0\}, \quad \phi_0 : U_0 \to \mathbb{C}, \quad [z_0 : z_1] \mapsto \frac{z_1}{z_0}, \quad [z_0 : z_1] = \left[1 : \frac{z_1}{z_0}\right]$$

$$U_1 = \{z_1 \neq 0\}, \quad \phi_1 : U_1 \to \mathbb{C}, \quad [z_0 : z_1] \mapsto \frac{z_0}{z_1}, \quad [z_0 : z_1] = \left[\frac{z_0}{z_1} : 1\right]$$

Note that $\phi_1 \circ \phi_0^{-1} : \mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto \frac{1}{z}$. We deduce that $\mathbb{P}_1(\mathbb{C}) \simeq \hat{\mathbb{C}}$, the identification given by

$$[1 : z] \mapsto z \in \mathbb{C}$$
$$[0 : 1] \mapsto \infty$$